Superconductivity by long-range color magnetic interaction in high-density quark matter*

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Abstract

We argue that in quark matter at high densities, the color magnetic field remains unscreened and leads to the phenomenon of color superconductivity. Using the renormalization group near the Fermi surface, we find that the long-range nature of the magnetic interaction changes the asymptotic behavior of the gap Δ at large chemical potential μ qualitatively. We find $\Delta \sim \mu g^{-5} \exp(-\frac{3\pi^2}{\sqrt{2}} \cdot \frac{1}{g})$, where g is the small gauge coupling. We discuss the possibility of breaking rotational symmetry by the formation of a condensate with nonzero angular momentum, as well as interesting parallels to some condensed matter systems with long-range forces.

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I. INTRODUCTION

The study of QCD at finite density has a long history. The suggestion that at high densities hadronic matter becomes quark matter [1] was made almost immediately after the discovery of asymptotic freedom [2]. It has been known for almost as long that at very high baryon densities, where perturbative QCD can be applied, quark matter is in a BCS-type superconducting state [3]. More recently, various groups [4,5] have modeled quark matter at intermediate densities using phenomenological four-fermion interactions and found that condensates form with sizable gaps of order 100 MeV. These studies reveal a rich phenomenology [4–8], including an interesting phase diagram [6], new symmetry breaking schemes like color-flavor locking [7], which might be also relevant for the study of nuclear matter in the nuclear superfluid phase, if the latter is continuously connected to the quark matter phase [8].

Naturally, a systematic calculation in perturbation theory is possible only in the high-density regime, where the chemical potential μ is much larger than Λ_{QCD} and the strong coupling at the scale of the Fermi energy is small. However, even in this case reliable calculations have so far been hampered by the inability to take into account the long-range nature of the color magnetic force, which dominates the interaction between the quarks at large distances. Instead of tackling the problem of condensate formation by a long-range interaction, many treatments rely on the assumption that a magnetic mass of order $g^2\mu$ is somehow generated. However, in reality, the magnetic field remains unscreened in the absence of superconductivity itself. Ordinary BCS theory cannot be directly applied due to the IR divergence from the exchange of the soft magnetic gluons. Therefore, even the asymptotic, weak-coupling behavior of the gap has not been found.

In this paper we present what we believe to be the correct estimate for the value of the BCS gap at asymptotically high densities. Our approach is based on the renormalization group around the Fermi surface [9,10], properly modified to take into account the long-range magnetic interaction. We find that the gap is proportional to $\exp(-c/g)$, $c = \frac{3\pi^2}{\sqrt{2}}$.[1] This behavior is different from the naive expectation from BCS theory, which predicts $\exp(-c/g^2)$. The fact that the suppression is parametrically much milder means a larger value of the gap at high densities, and potentially could also indicate that at intermediate densities the gap and the critical temperature may be larger than previously estimated. The latter may substantially enhance the chance of forming a color superconductor in heavy-ion collisions. We also show that the asymptotic behavior of the gap does not depend on the angular momentum of the Cooper pairs, and comment on the possibility of breaking rotational symmetry by the formation of a condensate with a nonzero angular momentum.

We will first review the renormalization group approach to the BCS theory (Sec. II), then describe the trouble caused by the long-range magnetic interaction (Sec. III). In Sec. IV we describe our resolution of this problem. We then make final remarks in Sec. V. The Appendices contain various technical details, including a treatment of Eliashberg theory.

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[1]This agrees with a comment in a recent paper by Pisarski and Rischke [11] that $\Delta \sim \exp(-c/g)$ for some constant c.

II. THE RENORMALIZATION GROUP NEAR THE FERMI SURFACE

An elegant method to see the formation of the BCS superconducting state is the renormalization group (RG) near the Fermi surface [9]. This approach has been applied to the case of quark matter by Evans, Hsu and Schwetz, and Schäfer and Wilczek [10]. These treatments apply when there exists a nonzero magnetic mass, screening the color magnetic interaction. For completeness, we give here an elementary overview of the approach, in the spirit of non-relativistic quantum mechanics. Readers who need a more formal, rigorous treatment should consult Ref. [10]. In this section we will follow Refs. [4,5] and consider a theory of quarks interacting via a local four-fermion interaction. The aim is to give an overview of the conventional RG approach before tackling the more difficult problem of a long-range interaction. Keeping in mind that one-gluon exchange conserves helicity, we will for simplicity consider only left-handed fermions.

Let us imagine a Fermi gas of massless quarks with a chemical potential μ. In the ideal gas approximation, all energy levels below the Fermi surface $|\mathbf{p}| = \mu$ are filled. The energy will be measured relative to the Fermi surface, so we introduce $\epsilon_{\mathbf{p}} = p - \mu$.

The RG procedure works as follows. At any given step, the effective theory contains only the fermion degrees of freedom located in a thin shell surrounding the Fermi surface. These fermions have $|\epsilon_{\mathbf{p}}| < \delta$. All others have been integrated out. The only relevant interaction between the fermions is the scattering of pairs with opposite momenta [9,10]. Let us introduce the scattering amplitude from a pair with momenta $(\mathbf{p}, -\mathbf{p})$ to another pair with momenta $(\mathbf{k}, -\mathbf{k})$,

$$f(\theta) \equiv f(\mathbf{p}, \mathbf{k}) = T(\mathbf{p}, -\mathbf{p} \to \mathbf{k}, -\mathbf{k})$$

To avoid complications with statistics, we will assume that the two particles are of different flavors. Near the Fermi surface, the amplitude depends only on the angle θ between $\mathbf{p}$ and $\mathbf{k}$. A positive f corresponds to a repulsive interaction, and a negative f means attraction between particles with opposite momenta.

In the spirit of the Wilson renormalization group, let us now integrate out all fermion states with $e^{-1}\delta < |\epsilon_{\mathbf{p}}| < \delta$. According to quantum mechanics, the scattering through virtual states in this region gives a correction to the scattering amplitude. To account for these virtual processes, we need to correct the scattering amplitude. Thus f obtains a correction,

$$f(\mathbf{p}, \mathbf{k}) \to f(\mathbf{p}, \mathbf{k}) - \sum_i \frac{T(\mathbf{p}, -\mathbf{p} \to i) T(i \to \mathbf{k}, -\mathbf{k})}{E_i - 2\epsilon_{\mathbf{p}}} \tag{1}$$

where the sum is over all intermediate states i belonging to the sector of the theory that has been integrated out. The virtual state i may have a different energy E_i than the initial energy $2\epsilon_{\mathbf{p}}$. We assume that the initial and final particles are almost exactly located at the Fermi surface, so $\epsilon_{\mathbf{p}} = \epsilon_{\mathbf{k}} = 0$. What could be the states i? To answer this question one notices that the scattering through an intermediate state can be of two types:

1. The pair $(\mathbf{p}, -\mathbf{p})$ can scatter to an intermediate pair $(\mathbf{p}', -\mathbf{p}')$, which then goes to $(\mathbf{k}, -\mathbf{k})$. In this case, the intermediate state i is that with two particle excitations with momenta $\pm\mathbf{p}'$. The Pauli principle requires that $\mathbf{p}'$ is located above the Fermi surface. This state has $E_i = 2\epsilon_{\mathbf{p}'}$ and $T(\mathbf{p}, -\mathbf{p} \to i) = f(\mathbf{p}, \mathbf{p}')$, $T(i \to \mathbf{k}, -\mathbf{k}) = f(\mathbf{p}', \mathbf{k})$.

2. Alternatively, first a pair of particles inside the Fermi sea with momenta $(\mathbf{p}', -\mathbf{p}')$ can scatter to make the final pair $(\mathbf{k}, -\mathbf{k})$, and then the initial pair $(\mathbf{p}, -\mathbf{p})$ scatters to fill the holes vacated by the pair $(\mathbf{p}', -\mathbf{p}')$ in the Fermi sphere. In this case, the intermediate state i consists of six elementary excitations: four particles with momenta $\pm\mathbf{p}$ and $\pm\mathbf{k}$, and two holes with momenta $\pm\mathbf{p}'$ located below the Fermi surface, $p' < \mu$. In this case, $E_i = -2\epsilon_{\mathbf{p}'}$, $T(\mathbf{p}, -\mathbf{p} \to i) = f(\mathbf{p}', \mathbf{k})$, $T(i \to \mathbf{k}, -\mathbf{k}) = f(\mathbf{p}, \mathbf{p}')$.

It is clear now that Eq. (1) becomes

$$f(\mathbf{p}, \mathbf{k}) \to f(\mathbf{p}, \mathbf{k}) - \int_{\mathbf{p}'} \frac{f(\mathbf{p}, \mathbf{p}') f(\mathbf{p}', \mathbf{k})}{2|\epsilon_{\mathbf{p}'}|} \tag{2}$$

where the integration is over all $\mathbf{p}'$ satisfying $e^{-1}\delta < |p - \mu| < \delta$. The integral over $|\mathbf{p}'|$ can be taken, and Eq. (2) reads,

$$f(\mathbf{p}, \mathbf{k}) \to f(\mathbf{p}, \mathbf{k}) - \frac{\mu^2}{2\pi^2} \int \frac{d\hat{\mathbf{p}}'}{4\pi} f(\mathbf{p}, \mathbf{p}') f(\mathbf{p}', \mathbf{k})$$

where the integration is over all directions of $\mathbf{p}'$. Repeating the RG procedure many times, one finds that f evolves according to the RG equation,

$$\frac{d}{dt} f(\mathbf{p}, \mathbf{k}) = -\frac{\mu^2}{2\pi^2} \int \frac{d\hat{\mathbf{p}}'}{4\pi} f(\mathbf{p}, \mathbf{p}') f(\mathbf{p}', \mathbf{k}) \tag{3}$$

where $t = -\ln\delta$ goes to $+\infty$ as one approaches the Fermi surface. Eq. (3) describes the RG evolution of the scattering amplitude on the Fermi surface.

It is convenient to expand the scattering amplitude over partial waves,

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l P_l(\cos\theta)$$

or, inversely, $f_l = \frac{1}{2}\int_0^\pi d\theta\, \sin\theta\, P_l(\cos\theta) f(\theta)$. Using a well known property of the Legendre polynomials, we find that the partial-wave amplitudes f_l evolve independently,

$$\frac{df_l}{dt} = -\frac{\mu^2}{2\pi^2} f_l^2 \tag{4}$$

The solution to Eq. (4) is

$$f_l(t) = \frac{f_l(0)}{1 + \frac{\mu^2}{2\pi^2} f_l(0) t}$$

We see that if at $t = 0$ all $f_l > 0$, which means that the interaction is repulsive in all channels, then the four-fermion interaction vanishes at the Fermi surface. However, if one of $f_l(0)$ is negative, it will develop a singularity (the Landau pole) at $t = -\frac{2\pi^2}{\mu^2 f(0)}$. The Landau pole is reached first by the channel having the largest negative $f_l(0)$. This singularity is nothing but the manifestation of the BCS instability of the Fermi surface with respect to any attractive interaction. The BCS gap is proportional to the energy scale at which the Landau pole is reached,

$$\Delta \sim \exp\Big(-\frac{2\pi^2}{\mu^2 f(0)}\Big)$$

Let us reproduce some results obtained in Ref. [10]. Let us take the interaction in the form $G^0(\bar\psi\gamma^0\psi)^2 + G^i(\bar\psi\gamma^i\psi)^2$, where G^0 and G^i are two independent constants. The tree-level scattering amplitude $\mathbf{p}, -\mathbf{p} \to \mathbf{k}, -\mathbf{k}$ between two left-handed particles arising from this interaction is

$$f(\theta) = 2\Big[G^0\cos^2\frac{\theta}{2} - G^i\Big(\cos^2\frac{\theta}{2} + 2\sin^2\frac{\theta}{2}\Big)\Big] \tag{5}$$

which contains only the s-wave and p-wave terms, with $f_0 = G^0 - 3G^i$ and $f_1 = \frac{1}{3}(G^0 + G^i)$. Therefore, G^0 and G^i can be said to run according to the following equations,

$$\begin{aligned}\frac{d}{dt}(G^0 - 3G^i) &= -\frac{\mu^2}{2\pi^2}(G^0 - 3G^i)^2\\ \frac{d}{dt}(G^0 + G^i) &= -\frac{\mu^2}{6\pi^2}(G^0 + G^i)^2\end{aligned}$$

which constitute a subset of the equations found in Ref. [10].

As a toy model mimicking the real one-gluon exchange, one can follow Ref. [7] and take the interaction of the form $-\frac{g^2}{3\Lambda^2}(\bar\psi\gamma_\mu\psi)^2$. Here Λ should be thought of as the typical momentum of the exchanged gluon, and $\frac{g^2}{3}$ is the effective coupling in the color $\bar{\mathbf{3}}$ channel [3,10], where the superconductivity effect is usually strongest. This corresponds to $G^0 = -G^i = -\frac{g^2}{3\Lambda^2}$. The interaction is most attractive in the s-channel and a BCS state is formed with the gap

$$\Delta \sim \exp\Big(-\frac{3\pi^2\Lambda^2}{2\mu^2 g^2}\Big) \tag{6}$$

The gap has the e^{-c/g^2} dependence on the coupling g. This parametric dependence of the gap has been obtained in Ref. [10] and is the same as that obtained in variational or mean field treatments of models with a four-fermion interaction between quarks [4–8]. We will see that in the true theory, the dependence at asymptotically high densities ($g \to 0$) is different from that obtained in these toy models.

III. THE PROBLEM OF THE UNSCREENED MAGNETIC FIELD

Let us try to naively apply the method developed in Sec. II to high-density QCD. At the lowest order, the quarks interact by exchanging one gluon. The gluon propagator, which is $1/q^2$ in vacuum, is modified by screening effects. The static color electric field is subjected to Debye screening at the distance scale m_{D}^{-1}, where $m_{\mathrm{D}} \sim g\mu$, which can be seen by resumming bubble diagrams in the gluon propagator. In the magnetic sector, the same resummation yields a magnetic gluon propagator,

$$D(q_0, q) = \frac{1}{q^2 + \frac{\pi}{2}m_{\mathrm{D}}^2\frac{|q_0|}{q}} \tag{7}$$

in the regime $q_0 \ll q \ll \mu$. The term $\frac{\pi}{2} m_{\rm D}^2 \frac{|q_0|}{q}$ comes from the Landau damping. In the static limit $q_0 = 0$, the magnetic field is not screened. If $q_0 \neq 0$, the field is said to be "dynamically screened" on the scale $q \sim m_{\rm D}^{2/3} q_0^{1/3}$.

Before going further, let us make an important comment on a confusion in the literature. This confusion originates from the similarity between the high-density and high-temperature gauge theories. The similarity can be seen in the Debye screening, which occurs at the scale gT at high temperatures and $g\mu$ at high densities. At high temperatures, the magnetic field is not screened at the one-loop level, but develops a non-perturbative screening of order g^2T. This sometimes leads to an unjustified assumption that the magnetic field develops a magnetic mass of order $g^2\mu$ in high-density QCD.

To see why the analogy between high temperatures and high densities breaks down on the question of the magnetic screening, let us review three standard ways to interpret the emergence of the magnetic mass in hot gauge theories. The first argument is dimensional reduction: the static large-distance behavior of a gauge theory at high temperatures is the same as of an effective Euclidean three-dimensional gauge theory. The latter is confined on the scale g^2T, which means that the magnetic field in the original theory should also be confined at this scale. The second argument is that the high-temperature perturbation theory is IR divergent for momenta $\lesssim g^2T$, and only a magnetic mass of this order could make the perturbation theory finite again. The third way is to notice that, due to the Bose enhancement, the thermal fluctuations of the gauge field become so large at the scale g^2T that they are fully non-linear. The last argument does not necessarily imply magnetic screening; it just shows that the existence of the latter at the scale g^2T does not contradict perturbative results, since at this scale the physics is non-perturbative.

None of these three arguments can be carried over the case of high densities. First, there is no dimensional reduction for gauge theories with a finite chemical potential. Second, the perturbation theory for the long-range magnetic field is infrared finite. Indeed, in vacuum, the infrared divergences of one-loop graphs come from integrals like $\int d^4q D^2(q)$, where $D(q) \sim q^{-2}$ is the gluon propagator. At finite μ, the gluon propagator is modified as in Eq. (7). Now q_0 effectively scales like q^3 instead of q, and by a simple power counting one sees that the integral is finite in the IR. Therefore perturbation theory does not break down for momenta of order $g^2\mu$, or $g^n\mu$ with any n, and there is no reason to expect non-perturbative effects proportional to any finite power of g.[2] The third argument explicitly relies on the large Bose enhancement that takes place only at finite temperatures; this argument clearly does not work at finite densities. Therefore, the magnetic interaction is not screened at the scale $g^2\mu$ as is occasionally assumed.

Now let us return to the our problem and try to apply the formalism developed in Sec. II to the interaction mediated by one-gluon exchange. We will see immediately that we have serious trouble with the very soft gluons. Indeed, on the Fermi surface, the tree-level small-angle ($\theta \ll 1$) scattering amplitude, due to one-gluon exchange, is

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[2]The author thanks S.Yu. Khlebnikov for pointing out this argument. Similar conclusion has been reached by Pisarski and Rischke [11].

$$f_{\rm tree}(\theta) = -\frac{2g^2}{3}\Big(\frac{1}{\mu^2\theta^2 + m_{\rm D}^2} + \frac{1}{\mu^2\theta^2}\Big) \tag{8}$$

The two contributions in the RHS come from the electric and the magnetic interaction, respectively (again, the factor $\frac{2}{3}$ comes from considering only the $\bar{\mathbf{3}}$ channel.) All partial amplitudes diverge logarithmically. For example,

$$f_0 = \frac{1}{2}\int\limits_{q_{\rm min}/\mu}^{\pi} d\theta\, \sin\theta\, P_l(\cos\theta)\, f(\theta) \approx -\frac{g^2}{3}\ln\frac{\mu}{q_{\rm min}} \tag{9}$$

where $q_{\rm min}$ is the smallest allowed momentum exchange that one has to put in by hand to make f_0 finite. Clearly, this IR problem renders the conventional RG formalism unusable.

Let us also warn against what might seem to be a natural solution to this IR problem. If one assumes that a quark condensate is eventually formed, the magnetic field is screened by the Meissner effect. One may be tempted to take the inverse London penetration length $g\Delta$ as the cutoff in Eq. (9), and use the computed value of f_0 to find the gap, thus obtaining a self-consistency condition for Δ. In this way one does obtain a gap of order $e^{-c/g}$, where c is some constant. However, this approach is flawed, and gives the wrong value of c, because it entirely neglects the screening effect of Landau damping which turns out to be much stronger than the Meissner effect. To see this, note that the condensate smears out the Fermi surface over a scale Δ, which is the natural energy scale of quasiparticles near the Fermi surface. The gluons that such excitations exchange have $q_0 \sim \Delta$. On these frequencies, the dynamical magnetic screening happens already at the scale $q \sim m_{\rm D}^{2/3}\Delta^{1/3}$, which is much larger than Δ. Therefore, the source of the IR cutoff should be the dynamical magnetic screening, which already takes place in the normal phase, rather than the Meissner effect.

Let us now turn to the central part of this paper, where we will find the correct RG treatment of the theory with the color magnetic interaction.

IV. RENORMALIZATION GROUP FOR MAGNETIC INTERACTIONS

Let us repeat the RG procedure described in Sec. II. At any given step in the RG evolution, one keeps only fermion modes having energies smaller than δ. The start of the RG evolution, $t=0$, will be taken at $\delta \sim m_{\rm D}$, and the evolution stops when δ is of order the gap, so typically $\delta \ll m_{\rm D}$.

At tree level, the fermions interact via one-gluon exchange, characterized by the momentum of the gluon $(q_0, \mathbf{q})$. Since all fermions have energy less than δ, the energy of the gluon q_0 is naturally of order or less than δ, while the momentum exchange q can be anywhere between 0 and 2μ.

Let us divide the four-fermion interaction that arises from the one-gluon exchange into "instantaneous" and "non-instantaneous" parts. The instantaneous interaction is mediated by the gluons that have momenta $q \gtrsim q_\delta \equiv m_{\rm D}^{2/3}\delta^{1/3}$. The Landau damping for these gluons is negligible, $m_{\rm D}^2\frac{|q_0|}{q} \lesssim q^2$. The gluon propagator, which is now simply q^{-2}, does not depend on q_0, which means that the four-fermion interaction they mediate can be considered as instantaneous. This part of the interaction is of the familiar type and will be treated in

the conventional way. In particular, one can characterize this part by the partial-wave amplitudes f_l. For $q \lesssim q_\delta$, the Landau damping can no longer be neglected. This part of the interaction has a considerable temporal retardation and should be treated separately.

Now let us reduce δ by a factor of $1/e$ by integrating out fermion degrees of freedom with energy between $e^{-1}\delta$ and δ. During this process the following will occur:

1. The partial-wave amplitudes f_l obtain the conventional renormalization, as written in Eq. (4).

2. One could ask if the non-instantaneous coupling is renormalized during this integration. To answer this question, one should compute the correction to the non-instantaneous interaction that comes from integrating out the fermion degrees of freedom. In Appendix A we explicitly estimate the corresponding one-loop diagrams and show that the non-instantaneous part of the interaction does not get renormalized.

3. Most importantly, and what makes our RG distinctive, *part of the non-instantaneous interaction becomes instantaneous.* Specifically, the gluon exchange with q lying in the interval $(e^{-1/3}q_\delta, q_\delta)$, which was formerly treated as non-instantaneous, now becomes a part of the instantaneous interaction and contributes to f_l. Simply speaking, our criterion of what to consider as instantaneous has became more inclusive, since we are now looking at a smaller energy scale, corresponding to a larger time scale.

How much of the non-instantaneous part of the interaction transfers to the instantaneous part during one step of the RG? According to Eq. (8) and the non-renormalization of the non-instantaneous interaction, the increment in $f(\theta)$ has the form,

$$\Delta f(\theta) = -\frac{2g^2}{3}\frac{1}{\mu^2\theta^2} \qquad \text{when } e^{-1/3}\frac{\delta}{\mu} < \theta < \frac{\delta}{\mu} \tag{10}$$

and vanishes outside this window of θ. For definiteness, let us concentrate our attention to the s-wave amplitude f_0. This amplitude obtains a constant additive contribution from the soft sector at each RG step,

$$\Delta f_0 = \frac{1}{2}\int\limits_{e^{-1/3}\delta\mu^{-1}}^{\delta\mu^{-1}} d\theta\,\sin\theta\,\Delta f(\theta) = -\frac{g^2}{9\mu^2}$$

Therefore, the RG group equation for f_0 now becomes,

$$\frac{d}{dt}f_0 = -\frac{g^2}{9\mu^2} - \frac{\mu^2}{2\pi^2}f_0^2 \tag{11}$$

The second term in the RHS is the familiar term that gives rise to the BCS effect for short-range interactions. What is new is the first term, which takes into account the fact that softer and softer gluon exchanges contribute to f_l. The non-instantaneous part of the interaction can be considered as an infinite pool, which continuously replenishes the instantaneous part during the RG evolution. Clearly, this should speed up the approach to the Landau pole.

To secure a solution we also need to specify an initial condition on f_0. Recall that $t = 0$ corresponds to $\delta \sim m_{\mathrm{D}}$, from Eq. (8) one finds, to the leading logarithm,

$$f_0(0) = -\frac{2g^2}{3\mu^2}\ln\frac{1}{g} \tag{12}$$

The solution to Eq. (11) with the initial condition (12) is

$$f_0(t) = -\frac{\sqrt{2}\pi g}{3\mu^2}\tan\Big[\frac{g}{3\sqrt{2}\pi}\Big(t + 6\ln\frac{1}{g}\Big)\Big]$$

The coupling f_0 hits the Landau pole when the argument of the tangent is equal $\frac{\pi}{2}$. This happens when

$$t = \frac{3\pi^2}{\sqrt{2}g} - 6\ln\frac{1}{g}$$

The Fermi liquid description, thus, breaks down at the energy scale

$$\Delta \sim m_{\mathrm{D}} e^{-t} \sim \mu g^{-5}\exp\Big(-\frac{3\pi^2}{\sqrt{2}g}\Big) \tag{13}$$

which will be interpreted as the scale of the gap. Notice that the gap is proportional to $e^{-c/g}$, which is parametrically larger than the naive estimate e^{-c/g^2} at small g. The reason for this enhancement is obviously the singularity of the magnetic interaction.

Strictly speaking, the RG calculation does not tell us that Δ is the value of the gap. In fact, the RG merely indicates that the normal Fermi liquid behavior breaks down at the scale of Δ. To confirm that Δ is the gap, one needs to use some alternative approach. In Appendix B, by making use of the Eliashberg equation, borrowed from the physics of electron-phonon systems, we verify that the gap Δ is in fact proportional to $\exp(-\frac{3\pi^2}{\sqrt{2}g})$.

V. REMARKS AND CONCLUSION

We have found that the ground state of the system of quarks, interacting via one-gluon exchange, is basically a BCS-type superconductor, and found the weak-coupling behavior of the gap using the RG approach, appropriate in the presence of a long-range magnetic interaction. We found $\Delta \sim g^{-5}e^{-c/g}$, $c = \frac{3\pi^2}{\sqrt{2}}$, not $\Delta \sim e^{-c/g^2}$ as in conventional BCS theory. Thus, at least in the $g \to 0$ limit, the superconductivity gap is larger than in all previous estimates [3–8,10]. Here let us make a few remarks on our calculation.

Wave function renormalization and non-Fermi-liquid behaviors. A similar problem of fermions interacting via an unscreened $U(1)$ magnetic field has also generated considerable interest in condensed matter physics. The fermions may be the valence electrons in metals [12], in which case the $U(1)$ field is the magnetic component of real electromagnetism, or some effective degrees of freedom in low-dimension strongly correlated electron systems [13], where the $U(1)$ gauge interaction could be generated as an effective interaction. In metals, the magnetic interaction is repulsive for a pair located on the opposite sides of the Fermi surface, so it does not lead to BCS superconductivity. However, interesting phenomena may still arise from this repulsive interaction. In particular, in has been shown [12] that the weak magnetic interaction leads to the breakdown of the Landau theory of Fermi liquid, typically at extremely low temperatures. One could ask whether the effects leading to this non-Fermi liquid behavior would modify our calculations.

The basic idea is that the fermion wave function gets a large renormalization near the Fermi surface from the magnetic interaction. To one-loop level, the renormalization of the wave function is [12]

$$Z^{-1}(q_0) = 1 + \text{const} \cdot g^2 \ln \frac{\mu}{q_0} \tag{14}$$

If one goes arbitrarily close to the Fermi surface, the wave function renormalization Z tends to 0, which means that the discontinuity of the distribution function at the Fermi surface disappears, thus signaling a deviation from Landau's Fermi liquid theory. However, in our case, the BCS effect is already essential at $q_0 \sim \Delta$. Recalling that $\Delta \sim e^{-c/g}$, we find from Eq. (14) that Z remains close to 1, $Z = 1 + O(g)$. Therefore, one can safely ignore the renormalization of the wave function, and there is no chance for non-Fermi-liquid behavior other than BCS-type superconductivity to manifest itself.

We note here that it was suggested that an attractive magnetic-type interaction may arise in various situations in condensed matter physics, for example in double-layer electron systems. A BCS gap may emerge in such systems. Due to a stronger singularity of the magnetic interaction in 2d, the gap is found to be proportional to a power of the coupling constant, rather than being exponential [14].

The very soft magnetic gluons. In our RG approach, even after the final step of the RG evolution, the interaction still contains a non-instantaneous sector, which is carried by the magnetic gluons with energy $q_0 \lesssim \Delta$ and momentum $q \lesssim m_{\mathrm{D}}^{2/3}\Delta^{1/3}$. The BCS effect is due to the instantaneous interaction, but one may ask whether the remaining non-instantaneous interaction could destroy the BCS state. Here we give an (admittedly crude) argument as to why this cannot happen.

The magnetic modes mediating the non-instantaneous interaction have their intrinsic time scale $q_0^{-1} \gtrsim \Delta^{-1}$. Therefore, these modes can be considered as static during the typical time scale of the system, Δ^{-1}. The question is now whether this random, static magnetic field could destroy the superconducting state. Let us estimate the total strength of the quantum fluctuations of the magnetic field. It is roughly

$$B^2 \sim \int^{q_\delta} d\mathbf{q}\, q^2 |A(\mathbf{q})|^2 \sim \int dq_0\, d\mathbf{q}\, q^2 \frac{1}{q^2 + \frac{\pi}{2} m_{\mathrm{D}}^2 \frac{|q_0|}{q}} \sim m_{\mathrm{D}}^2 \Delta^2$$

so the typical value of the fluctuating magnetic field is $m_{\mathrm{D}}\Delta \sim g\mu\Delta$. This should be compared with the critical magnetic field, which is of order $\mu\Delta$. We conclude that at weak coupling, the almost static quantum fluctuations of the magnetic field are too small to destroy the superconducting state.

The possibility of breaking rotational symmetry. In our treatment of the RG equation, we have concentrated our attention on the s-wave coupling f_0. However, we could equally consider higher partial waves. Let us take arbitrary l and try to rewrite Eq. (11) for f_l. According to Eq. (4), the f^2 part in the evolution of f_l has the same coefficient as for f_0. The constant part depends on how much the non-instantaneous sector throws out to the l-channel of the instantaneous sector at each step of the RG evolution. To find out, one should make the partial-wave expansion of the function Δf in Eq. (10). Since $\Delta f(\theta)$ is

concentrated on values of θ near 0, the partial-wave coefficients almost do not depend on l, provided that l is not parametric on g. Indeed,

$$\Delta f_l = \frac{1}{2} \int_{e^{-1/3}\delta\mu^{-1}}^{\delta\mu^{-1}} d\theta \, \sin\theta \, P_l(\cos\theta) \, \Delta f(\theta) \approx \frac{g^2}{9\mu^2} P_l(1) = \frac{g^2}{9\mu^2}$$

since the P_l are normalized so that $P_l(1) = 1$. Therefore, the RG equation for f_l is identical to that for f_0. The initial value for f_l is also independent of l, since it comes from the partial-wave expansion of the function $f_{\text{tree}}(\theta)$ in Eq. (8) which also peaks near $\theta = 0$. Therefore, to leading order, the RG does not discriminate between channels with different angular momenta. If a condensate with nonzero angular momentum is formed, the rotational symmetry is broken, like in the A phase of He^3.

However, at any finite g, the coincidence of the RG evolution of f_l with different l is not exact. Moreover, the RG approach does not give us the value of the gap, or the energy of the ground state, but merely yields the typical scale of the gap. Therefore, the two gaps with different l may have the same asymptotic behavior, but with different numerical coefficients, and the corresponding superconducting states may have different energies. At this stage, the natural assumption seems to be that the state with $l = 0$ is favored, and the ground state does not break rotational symmetry, but the question of forming a condensate with nonzero angular momentum of the Cooper pair should be investigated in a more careful manner. We defer this question to future work. We have also left out the possibility of a numerical estimation of the gap at moderate densities, which is a very interesting question from the phenomenological point of view. Presumably, a reasonable estimation could be found by solving the Eliashberg equation of the type described in Appendix B. Nor did we try to solve the problem at finite temperatures [6].

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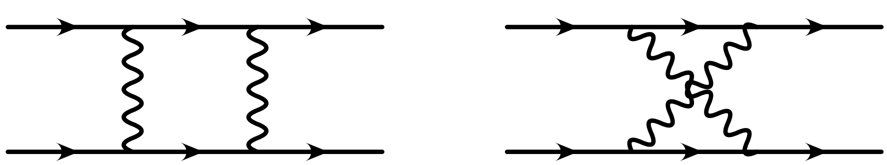
APPENDIX A: NON-RENORMALIZATION OF THE NON-INSTANTANEOUS INTERACTION

Here we present an explicit check that the soft sector of the theory is not renormalized during the RG flow. Let us consider one-loop corrections to the scattering amplitude $\mathbf{p}, -\mathbf{p} \to \mathbf{p}, -\mathbf{p}$. There are two Feynman diagrams to be considered:



The contributions from the two graphs are of the same order, let us evaluate only the first graph, which is of order

$$g^4 \int dk_0\, d\mathbf{k}\, \frac{1}{k_0^2+\epsilon_{\mathbf{p}+\mathbf{k}}^2} \cdot \frac{1}{(k^2+\frac{\pi}{2}m_{\mathrm{D}}^2\frac{|k_0|}{k})^2} \tag{A1}$$

Notice that the momentum of the internal fermion lines should be inside the shell $e^{-1}\delta < |\epsilon_{\mathbf{p}+\mathbf{q}}| < \delta$. Both gluon lines are supposed to belong to the non-instantaneous sector, so $k \lesssim m_{\mathrm{D}}^{2/3}\delta^{1/3}$. The integral over k_0 is dominated by $k_0 \lesssim \delta$. Therefore, the integral in Eq. (A1) is of order

$$g^4 \cdot \delta \cdot q_\delta^2 \delta \cdot \frac{1}{\delta^2} \cdot \frac{1}{q_\delta^4} \sim \frac{g^4}{q_\delta^2}$$

Therefore, the correction is of order $g^4 q_\delta^{-2}$.

Now assume that the external momenta of the final particles are slightly different from those of the initial particles, and the difference is q. The tree-level amplitude is of order $g^2 q^{-2}$. During each step of the RG evolution, this amplitude receives a correction of order $g^4 q_\delta^{-2}$, where δ is the moving RG scale. During the part of the RG evolution when the interaction is non-instantaneous, $q_\delta \gtrsim q$, and one sees that all the accumulated correction ($\sim g^4 q^{-2}$) is still smaller than the tree amplitude by a factor of g^2. One concludes therefore that there is no substantial renormalization of the non-instantaneous interaction during the RG evolution.

APPENDIX B: THE ELIASHBERG THEORY

The Eliashberg equations [15] can be considered as the generalization of the BCS gap equation to the case of a non-local interaction. Here we will be trying to reproduce only the leading exponential behavior $\exp(-\frac{3\pi^2}{\sqrt{2}g})$ of the gap, but not the power (g^{-5}) part. Presumably, a more careful treatment of the Eliashberg equation should reproduce the subleading g^{-5} factor and give an estimate for the numerical coefficient.

The generalization of the gap equation of the type written in Ref. [7] to the case of the non-local magnetic interaction is[3]

$$\Delta(p_0) = \frac{2}{3}g^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{|\mathbf{p}-\mathbf{q}|^2 + \mu^2 \frac{|p_0-q_0|}{|\mathbf{p}-\mathbf{q}|}} \cdot \frac{\Delta(q_0)}{q_0^2+\epsilon_{\mathbf{q}}^2+\Delta^2(q_0)} \tag{B1}$$

[3] The coefficient $\frac{2}{3}$ in Eq. (B1) can be understood as follows. If one replaces the gluon propagator in Eq. (B1) by $\frac{1}{\Lambda^2}$, Eq. (B1) gives the same gap as Eq. (6) if the numerical coefficient in the RHS is $\frac{4}{3}$. The coefficient one should put in the RHS of Eq. (B1) should be twice smaller than that, due to the screening of the electric field, which reduces the effective coupling by a factor of 2. For the color-flavor locking scheme [7], there should be two equation for two gaps; in weak coupling they can be written as one equation (B1) which is valid to the leading order.

Notice that, as in the usual Eliashberg theory, the gap is a function of p_0 only but not of $\mathbf{p}$. Indeed, the dependence of the RHS of Eq. (B1) on $\mathbf{p}$ is only in the gluon propagator, and if $\mathbf{p}$ is near the Fermi surface any change of $\mathbf{p}$ can be compensated by a rotation of $\mathbf{q}$. The conventional Eliashberg equations are actually a set of equations for $\Delta(p_0)$ and $Z(p_0)$, where $Z(p_0)$ is the wave function renormalization, but as explained in Sec. V, $Z(p_0) \approx 1$. Notice also that in Eq. (B1) we write μ^2 instead of $\frac{\pi}{2}m_{\rm D}^2$, the reason is that we will be working with exponential scales so the difference between these two coefficients will not affect the leading exponent. Integrating over $\mathbf{q}$, one finds,

$$\Delta(p_0) = \frac{g^2}{18\pi^2}\int\limits_0^\infty dq_0 \, \ln\frac{\mu}{|p_0-q_0|}\cdot\frac{\Delta(q_0)}{\sqrt{q_0^2+\Delta^2(q_0)}} \tag{B2}$$

where only the leading logarithm is written. To the leading log, the integral in Eq. (B2) can be split into two regions, $0<q_0<p_0$ and $p_0<q_0<\mu$, where in the first $\ln\frac{\mu}{|p_0-q_0|}\approx\ln\frac{\mu}{p_0}$ and in the second $\ln\frac{\mu}{|p_0-q_0|}\approx\ln\frac{\mu}{q_0}$. Introducing the logarithmic scales

$$x=\ln\frac{\mu}{p_0},\quad y=\ln\frac{\mu}{q_0},\quad x_0=\ln\frac{\mu}{\Delta_0}$$

where $\Delta_0=\Delta(p_0=0)$, the Eliashberg equation becomes,

$$\Delta(x) = \frac{g^2}{18\pi^2}\Bigl(x\int\limits_x^{x_0}dy\,\Delta(y)+\int\limits_0^x dy\,y\Delta(y)\Bigr) \tag{B3}$$

Differentiating Eq. (B3) with respect to x twice, one finds,

$$\Delta''(x) = -\frac{g^2}{18\pi^2}\Delta(x) \tag{B4}$$

As the boundary conditions, from Eq. (B3) one can check that $\Delta(0)=0$ and $\Delta'(x_0)=0$. The solution to Eq. (B4) is,

$$\Delta(x)=\Delta_0\sin\Bigl(\frac{g}{3\sqrt{2}\pi}x\Bigr)$$

To satisfy the boundary condition at x_0, one requires $\frac{g}{3\sqrt{2}\pi}x_0=\frac{\pi}{2}$, from which one finds $x_0=\frac{3\pi^2}{\sqrt{2}g}$, and the gap at small energies is $\Delta_0\sim e^{-x_0}\sim\exp(-\frac{3\pi^2}{\sqrt{2}g})$, reproducing the leading exponential behavior of our RG result. This is the minimal energy cost to create a fermion excitation. The energy-dependent gap $\Delta(p_0)$ is

$$\Delta(p_0)=\Delta_0\sin\Bigl(\frac{g}{3\sqrt{2}\pi}\ln\frac{\mu}{p_0}\Bigr)$$

for $p_0\gtrsim\Delta_0\sim\exp(-\frac{3\pi^2}{\sqrt{2}g})$.

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